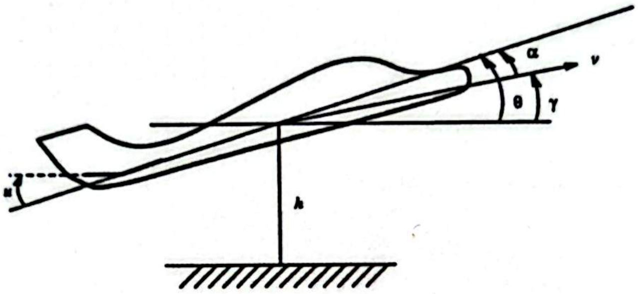
Linear systems – Midterm exam

Midterm exam 2025–2026, Wednesday 27 May, 18:30 – 20:30

Instructions

1. The use of books and lecture notes is not allowed, but you can use a one-page cheat sheet.
2. All answers need to be accompanied with an explanation or calculation.

Problem 1 (6 + 4 + 12 + 5 + 5 + 12 = 44 points)



The dynamics of an airplane are given, after appropriate scaling, by

$$\begin{aligned} \dot{\gamma}(t) &= \sin\big(\theta(t) - \gamma(t)\big), \\ \ddot{\theta}(t) &= -\theta(t) + \gamma(t) + u(t), \\ \dot{h}(t) &= \sin\gamma(t). \end{aligned} \tag{1}$$

Here, γ is the flight angle, θ is the angle between the reference axis of the airplane and the horizontal, and h is the flight altitude, see the figure. The elevator angle u is the control input.

(a) Write the equations of motion (1) in state-space form (i.e., as a system of first-order differential equations) by choosing the state $x = [\gamma\ \theta\ \dot{\theta}\ h]^\top$.

(b) Show that, for any $\bar{h} > 0$, $\bar{x} = [0\ 0\ 0\ \bar{h}]^\top$ is an equilibrium point for $\bar{u} = 0$ for the airplane model.

(c) Linearize the state-space model obtained in (a) around the equilibrium $(\bar{x}, \bar{u})$ given in (b), to obtain a linear system of the form

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t). \tag{2}$$

(d) Show that, for $\tilde{u} = 0$, the linearized system (2) is not asymptotically stable.

In an attempt to stabilize the linear system (2), we choose the input $\tilde{u}$ according to

$$\tilde{u}(t) = k\big(h(t) - \bar{h}\big) \quad \text{for some } k \in \mathbb{R}. \tag{3}$$

(e) Show that the controlled linearized system, i.e., (2) with controller (3), can be written as

$$\dot{\tilde{x}}(t) = A_k\tilde{x}(t) \tag{4}$$

for some matrix A_k (that depends on k). Make sure to give the matrix A_k.

(f) Show that, for any $k \in \mathbb{R}$, the controlled system (4) is not asymptotically stable. In other words, the controller (3) cannot stabilize the linearized system (2).

Note. In case you did not find the matrix A_k in (e), you can instead use

$$A_k = \begin{bmatrix} 0 & 0 & 0 & 1 \\ k & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Problem 2 (6 + 6 = 12 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \text{with} \qquad A = \begin{bmatrix} 2 & 1 & 1 \\ -5 & -3 & -4 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Is the linear system controllable?

(b) Give a basis for the reachable subspace.

Problem 3 (8 + 8 + 8 + 10 = 34 points)

Consider the *time-varying* linear system

$$\dot{x}(t) = A(t)x(t), \tag{5}$$

with state x of dimension n and where the system matrix depends (continuously) on time, i.e., $A : \mathbb{R} \to \mathbb{R}^{n\times n}$. Denote the set of all solutions to (5) by

$$\mathcal{S} = \left\{x \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n) \mid \dot{x}(t) = A(t)x(t) \text{ for all } t \in \mathbb{R}\right\},$$

where $\mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ represents the vector space of continuously differentiable functions $x : \mathbb{R} \to \mathbb{R}^n$.

(a) Show that $\mathcal{S}$ is a subspace (of $\mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$).

In the remainder of this problem, we consider the associated initial value problem

$$\dot{x}(t) = A(t)x(t), \qquad x(t_0) = x_0 \tag{6}$$

and denote the resulting solution (at time $t \in \mathbb{R}$ and for given (t_0, x_0)) as $x(t; t_0, x_0)$. You may assume that, for any (t_0, x_0), this solution is *unique*. However, contrary to the case where A is constant, there is no explicit expression for the solution to (6).

(b) Prove the following: if $x(\tau; t_0, x_0) = 0$ for some $\tau \in \mathbb{R}$, then $x(t; t_0, x_0) = 0$ for all $t \in \mathbb{R}$.

(c) Prove the following: for any $x_0^1, x_0^2 \in \mathbb{R}^n$ and any $c_1, c_2 \in \mathbb{R}$, we have

$$c_1 x(t; t_0, x_0^1) + c_2 x(t; t_0, x_0^2) = x\left(t; t_0, c_1 x_0^1 + c_2 x_0^2\right) \quad \text{for all } t.$$

(d) Prove the following: if the initial vectors x_0^1 and x_0^2 are linearly independent, then the vectors

$$x(t; t_0, x_0^1) \quad \text{and} \quad x(t; t_0, x_0^2)$$

are linearly independent for each $t \in \mathbb{R}$.

Hint. Use the statements in problems (b) and (c).

(10 points free)